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Stieltjes Moment Problems and the Friedrichs Extension of a Positive Definite Operator

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For an indeterminate Stieltjes moment sequence the multiplication operator $Mp(x) = xp(x)$ is positive definite and has self-adjoint extensions. Exactly one of these extensions has the same lower bound as M, the so-called Friedrichs extension. The spectral measure of this extension gives a certain solution to the moment problem and we identify the corresponding parameter value in the Nevanlinna parametrization of all solutions to the moment problem. In the case where σ is indeterminate in the sense of Stieltjes, relations between the (Nevanlinna matrices of) entire functions associated with the measures $t^k\, d\sigma(t)$ are derived. The growth of these entire functions is also investigated.

1. INTRODUCTION

We consider a normalized Hamburger moment sequence $(s_n)_{n \geq 0}$ and the corresponding polynomials $(P_k)_{k \geq 0}$, $(Q_k)_{k \geq 0}$ of the first and second kind, following the notation of Akhiezer, [1]. The sequence $(P_k)_{k \geq 0}$ forms an orthonormal system in $\mathbf{C}[\mathbf{x}]$ with the inner product given by $\langle x^n, x^m \rangle = \int_{-\infty}^{\infty} x^{n+m}\, d\mu(x)$, where μ is any measure from the set

$$V = \left\{ \mu \geq 0 \mid s_n = \int x^n\, d\mu(x),\ \forall n \geq 0 \right\} \tag{1}$$

of solutions to the moment problem. The P_k's are uniquely determined by the requirements that degree $P_k = k$, and that the leading coefficient is positive. The Q_k's are given by

$$Q_k(x) = \int_{-\infty}^{\infty} \frac{P_k(x) - P_k(t)}{x - t}\, d\mu(t), \tag{2}$$

where μ is any measure from V.

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The sequence $(s_n)_{n\geq 0}$ is called indeterminate if V contains more than one element. It turns out that the sequence is indeterminate if and only if

$$p(z)=\left(\sum_{k=0}^{\infty}|P_k(z)|^2\right)^{1/2} \tag{3}$$

is finite for some non-real z, in which case the series converges uniformly over compact subsets of the complex plane.

In the indeterminate case we have the Nevanlinna parametrization of the set V. To describe it, let $\mathscr{P}$ denote the set of analytic functions $\phi\colon \{\mathscr{I}z>0\}\to\{\mathscr{I}z\geq 0\}$. Any such function can be extended to a function defined on $\mathbf{C}\setminus\mathbf{R}$ by reflection in the real axis. The parametrization is the one-to-one correspondence between $\mathscr{P}\cup\{\infty\}$ and V given by $\phi\leftrightarrow\mu_\phi$, where

$$\int_{-\infty}^{\infty}\frac{1}{z-x}\,d\mu_\phi(x)=\frac{A(z)\,\phi(z)-C(z)}{B(z)\,\phi(z)-D(z)},\qquad \mathscr{I}z\neq 0. \tag{4}$$

The functions A, B, C, D are entire and they form the Nevanlinna matrix $\left(\begin{smallmatrix}A & C\\ B & D\end{smallmatrix}\right)$ associated with the moment problem, see Akhiezer, [1], p. 55 and Berg & Pedersen, [5]. They are defined by the corresponding polynomials

$$\begin{aligned}
A(z)&=z\sum Q_k(0)\,Q_k(z)\\
B(z)&=-1+z\sum Q_k(0)\,P_k(z)\\
C(z)&=1+z\sum P_k(0)\,Q_k(z)\\
D(z)&=z\sum P_k(0)\,P_k(z).
\end{aligned} \tag{5}$$

Another approach to the moment problem is via the theory of unbounded operators in Hilbert space. Let $\mathscr{H}$ be the completion of $\mathbf{C}[\mathbf{x}]$ in the above inner product. We consider the multiplication operator $M\colon \mathbf{C}[\mathbf{x}]\to\mathbf{C}[\mathbf{x}]$ given by $Mp(x)=xp(x)$. This is a densely defined symmetric operator in $\mathscr{H}$ and it turns out that it has self-adjoint extensions in $\mathscr{H}$, see eg. Akhiezer, [1], p. 140–141. The spectral measures of these self-adjoint operators give solutions to the moment problem. There is in fact a one-to-one correspondence between the set of solutions μ for which the polynomials are dense in $L^2(\mu)$, the so-called Nevanlinna-extremal solutions, and the set of self-adjoint extensions in $\mathscr{H}$ given by $\mu = \langle E(\cdot) P_0, P_0 \rangle$, where E is a spectral measure, see Stone, [17], Theorem 10.30, or Akhiezer, [1], p. 144.

To obtain all solutions to the moment problem one needs to consider self-adjoint extensions of M into Hilbert spaces where $\mathscr{H}$ is a closed subspace. Concerning this, see for example the survey by Sarason, [16].

In the Nevanlinna parametrization (4) we can identify the Nevanlinna-extremal measures as the set $(\mu_t)_{t \in \mathbf{R} \cup \{\infty\}}$ and so we parameterize the spectral measures as $(E^t)_{t \in \mathbf{R} \cup \{\infty\}}$, where $\mu_t = \langle E^t(\cdot) P_0, P_0 \rangle$.

In this paper we shall consider a Stieltjes moment sequence $(t_n)_{n \geq 0}$ which is indeterminate as a Hamburger moment sequence. In this case the operator M is positive definite. Among all self-adjoint extensions of M in $\mathscr{H}$ there is one with the same lower bound as M. This is called the Friedrichs extension of M. It is in general uniquely determined by the additional condition that its domain should be contained in the completion of $\mathbf{C}[\mathbf{x}]$ in a different inner product. Regarding the Friedrichs extension see e.g. [15], section 124. Our object is to determine the parameter value of the corresponding Nevanlinna-extremal measure μ_t in the Nevanlinna-parametrization.

Chihara has investigated the N-extremal solutions to the Hamburger moment problem for $(t_n)_{n \geq 0}$, see [10] and [11]. In [11] a certain N-extremal solution with largest least spectral point was found, and in [10] all Nevanlinna-extremal solutions which have their support in $[0; \infty)$ were identified. It was found that the corresponding parameter values form an interval $[\alpha; 0]$, where α is a certain non-positive number. We show that α is the parameter corresponding to the Friedrichs extension of M.

Let σ be a measure having moments of all orders with $\operatorname{supp}(\sigma) \subseteq [0; \infty)$. We shall say that σ is determinate (resp. indeterminate) in the sense of Stieltjes and write det(S) (resp. indet(S)) if there are no other (resp. other) measures on $[0; \infty)$ with the same moments as σ.

In the case where σ is indet(S) the measures $t^k \, d\sigma(t)$, $k \geq 1$ are also indet(S). We shall express the entire functions associated with $t^k \, d\sigma(t)$ in terms of the entire functions associated with σ. This is done by considering the symmetric Hamburger moment problem that corresponds to the Stieltjes moment problem for σ, as in Chihara, [10]. Relations between the growth of the different entire functions are found.

The transformation $\sigma \mapsto t \, d\sigma(t)$ has been investigated independently by Galliano Valent ([18]) in connection with birth and death processes in the indeterminate case. For a birth and death process with rates (λ_n), (μ_n) and corresponding Stieltjes moment problem (t_n) Karlin and McGregor consider a dual process having rates (μ_{n+1}), (λ_n). The Stieltjes moment problem corresponding to the dual process is (t_{n+1}/t_1), see [12] and [13].

2. Preliminaries about Operators

Let $(T, \mathscr{D}_T)$ be a self-adjoint operator in a Hilbert space $\mathscr{H}$. The lower bound of T is defined as

$$m_T = \inf\{\langle Tx, x\rangle \mid x \in \mathscr{D}_T, \|x\| = 1\}.$$

Let E be the spectral measure corresponding to T. The support of E, $\operatorname{supp}(E)$, is the smallest closed subset Ω of $\mathbf{R}$ such that $E(\Omega) = I$. We denote by $dE_{x,y}(t)$ the complex Borel measure $\omega \mapsto \langle E(\omega)x, y\rangle$. For the sake of completeness we include the following well-known result.

Lemma 2.1. $m_T = \inf(\operatorname{supp}(E))$.

Proof. For $x \in \mathscr{D}_T$ of norm 1 we have $\langle Tx, x\rangle = \int_{\operatorname{supp}(E)} t\, dE_{x,x}(t) \geqslant \inf \operatorname{supp}(E)$. Thus $m_T \geqslant \operatorname{supp}(E)$. On the other hand, if $t_0 \in \operatorname{supp}(E)$ and V_0 is any bounded open neighbourhood of t_0 we have $E(V_0) \neq 0$. There is thus $x_0 \in \operatorname{Ran} E(V_0)$ of norm 1. This implies

$$\begin{aligned} m_T &\leqslant \langle Tx_0, x_0\rangle = \langle TE(V_0)\,x_0, E(V_0)\,x_0\rangle \\ &= \int_{V_0} t\, dE_{x_0,x_0}(t) \leqslant \sup V_0, \end{aligned}$$

thence $m_T \leqslant t_0$ for all $t_0 \in \operatorname{supp}(E)$ and the lemma follows. ∎

A self-adjoint operator $(T, \mathscr{D}_T)$ with spectral measure E is said to have simple spectrum if there is x in $\mathscr{H}$ such that $\{E(\omega)\,x \mid \omega \in \mathbf{B}_1\}$ spans a dense subset of $\mathscr{H}$ (here $\mathbf{B}_1$ denotes the Borel sets of $\mathbf{R}$). Such an element x is called a cyclic vector. It is known that T has simple spectrum if and only if there is an element $x_0 \in \bigcap_{n\geqslant 1} \mathscr{D}_{T^n}$ such that $\{T^n x_0\}_{n\geqslant 0}$ spans a dense subspace of $\mathscr{H}$, in which case x_0 is also a cyclic vector. See e.g. [2], section 69. We need the following lemma.

Lemma 2.2. *Suppose that $(T, \mathscr{D}_T)$ has simple spectrum with cyclic vector x_0. Then*

$$\operatorname{supp}(E) = \operatorname{supp}(E_{x_0,x_0}).$$

Proof. If $t \notin \operatorname{supp}(E_{x_0,x_0})$ we can choose an open neighbourhood ω_0 of t such that $E_{x_0,x_0}(\omega_0) = 0$. Then for all Borel sets ω, $E_{x_0,x_0}(\omega \cap \omega_0) = 0$ and hence $E(\omega \cap \omega_0)\,x_0 = 0$. If $x \in \mathscr{H}$ then $\langle E(\omega_0)\,x, E(\omega)\,x_0\rangle = \langle x, E(\omega \cap \omega_0)\,x_0\rangle = 0$ and since x_0 is a cyclic vector we must have $E(\omega_0)\,x = 0$. Therefore $E(\omega_0) = 0$ and $t \notin \operatorname{supp} E$. The other inclusion is easy. ∎

Remark 2.1. If $(T, \mathscr{D}_T)$ is a self-adjoint extension of the multiplication operator M associated with the moment problem then T has simple spectrum with cyclic vector P_0.

3. The Parameter Corresponding to the Friedrichs Extension

Let $(t_n)_{n \geq 0}$ be a normalized Stieltjes moment sequence, which is indeterminate on the whole line. The full set of solutions on the line we denote by $(\mu_\phi)_{\phi \in \mathscr{P} \cup \{\infty\}}$, cf. (4). At least one solution σ has its support contained in $[0; \infty)$ and therefore the zeros of $(P_n)_{n \geq 0}$ and $(Q_n)_{n \geq 1}$ are all positive. In particular, $P_n(0) \neq 0$, $Q_n(0) \neq 0$. From Lemma 1, [11], we see that the sequence $P_n(0)/Q_n(0)$ is convergent, say

$$\frac{P_n(0)}{Q_n(0)} \xrightarrow[n]{} \alpha, \tag{6}$$

for some $\alpha \in (-\infty; 0]$. In fact $P_n(0)/Q_n(0) = -m_1^{-1}(\sum_{i=1}^n l_i)^{-1}$, where (m_i, l_i) are the coefficients in the corresponding Stieltjes continued fraction, see Berg, [3]. Defining $b_n = \int x P_n(x)\, P_{n+1}(x)\, d\sigma(x)$, $n \geq 0$, we consider the functions

$$\begin{aligned} B_{n+1}(z) &= b_n(Q_n(0)\, P_{n+1}(z) - Q_{n+1}(0)\, P_n(z)) \\ D_{n+1}(z) &= b_n(P_n(0)\, P_{n+1}(z) - P_{n+1}(0)\, P_n(z)). \end{aligned} \tag{7}$$

These functions converge to the entire functions B and D given in (5), and the convergence is uniform over compact sets of the complex plane. By (7) and [1], p. 9 we get

$$B_{n+1}(z) - \frac{Q_{n+1}(0)}{P_{n+1}(0)} D_{n+1}(z) = -\frac{P_{n+1}(z)}{P_{n+1}(0)}.$$

Letting n tend to ∞, we see that

$$\alpha B(z) - D(z) = -\lim_{n \to \infty} \frac{P_n(z)}{Q_n(0)}. \tag{8}$$

This equation is derived in Chihara, [10] or [11], using a different notation. For the readers convenience we have repeated the derivation here.

Proposition 3.1 (Chihara [11]). *If ν is any solution to the Hamburger moment problem for the Stieltjes moment sequence $(t_n)_{n \geq 0}$ then*

$\inf \operatorname{supp}(\nu) \leqslant \inf \operatorname{supp}(\mu_\alpha)$. *If equality holds here and ν is N-extremal then $\nu = \mu_\alpha$.*

Proof. Let x be any root of the entire function $\alpha B - D$. If $\varepsilon > 0$ is given, Rouchés theorem and (8) show that there is a root of some P_n in the interval $(x-\varepsilon; x+\varepsilon)$. Suppose that $\inf(\operatorname{supp}(\nu)) > \inf(\operatorname{supp}(\mu_\alpha)) = \inf Z(\alpha B - D)$. Since all zeros of the polynomials $(P_n)_{n \geqslant 0}$ belong to the open interval $(\inf \operatorname{supp}(\nu); \sup \operatorname{supp}(\nu))$, this contradicts the fact that the P_n's have zeros arbitrarily close to the smallest zero of $\alpha B - D$.

From (4) it follows that the supports of two different N-extremal measures are disjoint. This proves the proposition. ▮

PROPOSITION 3.2. *Let $(t_n)_{n \geqslant 0}$ be a normalized Stieltjes moment sequence which is indeterminate on the whole line. The parameter in the Nevanlinna parametrization (4) corresponding to the Friedrichs extension is given by*

$$\alpha = \lim_{n \to \infty} \frac{P_n(0)}{Q_n(0)}. \tag{9}$$

Proof. Let $(T_t)_{t \in \mathbf{R} \cup \{\infty\}}$ denote the self-adjoint extensions of the operator M in $\mathscr{H}$. By combining Lemmas 2.1 and 2.2 we see that the lower bound of T_t is the smallest zero of the entire function $tB - D$ ($= B$ for $t = \infty$). Thus by Proposition 3.1 the lower bound of T_α is strictly greater than the lower bound of any other operator T_t. Therefore T_α must be the Friedrichs extension of M. ▮

The parameter α of the Friedrichs extension can also be found without referring to (8). In Berg & Valent, [9], it is shown that the function D/B is strictly decreasing on each of the intervals $(-\infty; \beta_1)$, $(\beta_1; \beta_2)$, $\cdots$ with

$$\lim_{x \to -\infty} D(x)/B(x) = \alpha. \tag{10}$$

Here $(\beta_n)_{n \geqslant 1}$ are the zeros of B and they satisfy $\beta_1 < 0 < \beta_2 < \cdots$. Information about the smallest zero of $tB - D$ can then be obtained by considering the strictly increasing function $t - D/B = (tB - D)/B$, and by varying the parameter t. This method also identifies the N-extremal solutions having their support in $[0; \infty)$, see Remark 2.2.2 in [9].

We end this section by giving a generalization of Proposition 3.2 to moment problems on a half-line. Let μ be a positive measure on $\mathbf{R}$ with moments of all orders and corresponding polynomials P_n, Q_n. Consider the measure $\tau_c(\mu)$, where $\tau_c(x) = x - c$. The polynomials corresponding to $\tau_c(\mu)$ are $P_n(x+c)$, $Q_n(x+c)$. The map τ_c is a bijection $V \to V_c$ between the sets of solutions to the two moment problems.

Suppose that μ is indeterminate on $\mathbf{R}$. We can express the entire functions $A^{(c)}$, $B^{(c)}$, $C^{(c)}$, $D^{(c)}$ associated with $\tau_c(\mu)$ in terms of those associated with μ:

PROPOSITION 3.3.

$$A^{(c)}(z) = C(c)\,A(z+c) - A(c)\,C(z+c)$$
$$B^{(c)}(z) = C(c)\,B(z+c) - A(c)\,D(z+c)$$
$$C^{(c)}(z) = D(c)\,A(z+c) - B(c)\,C(z+c)$$
$$D^{(c)}(z) = D(c)\,B(z+c) - B(c)\,D(z+c).$$

Proof. We consider the two-variable analogues of the entire functions $A^{(c)}$, $B^{(c)}$, $C^{(c)}$, $D^{(c)}$, see [6], p. 175. We have $A^{(c)}(u, v) = A(u+c, v+c)$ and similarly for the other functions. The formulas then follow from Proposition 4.6, p. 177 in [6]. ∎

PROPOSITION 3.4. *The Nevanlinna parametrizations of the two sets V and V_c are related as*

$$(\sigma^{(c)})_\phi = \tau_c(\sigma_{\phi^{(c)}}),$$

where

$$\phi^{(c)}(z) = \frac{C(c)\,\phi(z-c) - D(c)}{A(c)\,\phi(z-c) - B(c)}.$$

Proof. By the above proposition and (4) we see that $\int (1/(t-z))\, d(\sigma^{(c)})_\phi(t) = \int (1/(t-z))\, d\tau_c(\sigma_{\phi^{(c)}})(t)$. ∎

Remark 3.1. The above linear fractional transformation $w \mapsto (C(c)\,w - D(c))/(A(c)\,w - B(c))$ maps the upper half-plane conformally onto itself.

PROPOSITION 3.5. *Suppose that $(s_n)_{n \geqslant 0}$ is an indeterminate moment sequence having a solution whose support is contained in $[c; \infty)$. Then the Friedrichs extension of our multiplication operator M corresponds to the parameter*

$$\frac{C(c)\lim(P_n(c)/Q_n(c)) - D(c)}{A(c)\lim(P_n(c)/Q_n(c)) - B(c)} \tag{11}$$

in the Nevanlinna parametrization of all solutions to the moment problem.

Proof. The operator M is bounded below, so the Friedrichs extension still exists. If $\operatorname{supp}(\mu)$ is contained in $[c;\infty)$, $\tau_c(\mu)$ is concentrated on $[0;\infty)$ and the parameter corresponding to the Friedrichs extension (for this translated problem) is then given by

$$\alpha = \lim_{n\to\infty} \frac{P_n(c)}{Q_n(c)}.$$

By Proposition 3.4, $(\sigma^{(c)})_\alpha = \tau_c(\sigma_{\alpha^{(c)}})$, where $\alpha^{(c)}$ is given as in (11). ∎

4. Symmetric Moment Problems

A normalized Hamburger moment sequence $(s_n)_{n\geq 0}$ is called symmetric if

$$s_{2n+1} = 0, \qquad \forall n \geq 0.$$

Symmetric moment problems have been investigated by Chihara, [10] and rotation invariant moment problems, a generalization to measures in $\mathbf{R}^n$, have been studied by Berg & Thill, [7]. There is a close relation between symmetric moment problems and Stieltjes moment problems.

Let $t=(t_n)_{n\geq 0}$ be a normalized Stieltjes moment sequence and define $s=(s_n)_{n\geq 0}=(t_0, 0, t_1, 0, t_2, \cdots)$. We say that $(s_n)_{n\geq 0}$ is the corresponding symmetric Hamburger moment problem. We denote by

$$W_t = \left\{\sigma \geq 0 \mid \operatorname{supp}(\sigma) \subseteq [0;\infty),\ t_n = \int x^n\, d\sigma(x),\ n\geq 0\right\}$$

$$V_s = \left\{\mu \geq 0 \mid s_n = \int x^n\, d\mu(x),\ n \geq 0\right\}$$

the sets of solutions to the two moment problems. Let the functions $\phi\colon \mathbf{R}\to[0;\infty)$, $\psi\colon[0;\infty)\to[0;\infty)$ be given by $\phi(x)=x^2$, $\psi(x)=\sqrt{x}$. Finally, if λ is a measure we put $\tilde{\lambda}(E)=\lambda(-E)$. The measure λ is called symmetric if $\tilde{\lambda}=\lambda$. With this notation we have

Proposition 4.1. *The map given by $\sigma \mapsto 1/2(\psi(\sigma)+\psi(\sigma)^{\sim})$ is a bijective mapping from W_t to the symmetric measures in V_s, whose inverse is given by $\mu\mapsto\phi(\mu)$.*

Proof. This is straightforward, except perhaps to show that

$$\tfrac{1}{2}(\psi(\phi(\mu))+\psi(\phi(\mu))^{\sim})=\mu. \tag{12}$$

Here one can write $\mu = \mu(\{0\})\,\delta_0 + \tau + \check{\tau}$, where τ is the restriction of μ to $(0;\infty)$. Then

$$\psi(\phi(\mu)) = 2\tau + \mu(\{0\})\,\delta_0,$$
$$\psi(\phi(\mu))\check{} = 2\check{\tau} + \mu(\{0\})\,\delta_0,$$

so that (12) holds. ∎

Denote by $(S_n)_{n\geq 0}$, $(S_n^{(1)})_{n\geq 0}$ the polynomials of the first and second kind corresponding to the symmetric Hamburger moment sequence $(s_n)_{n\geq 0}$. We note that $S_n(-x) = (-1)^n S_n(x)$ so that S_{2n} is an even function and S_{2n+1} is odd. By (2), $S_{2n}^{(1)}$ is odd and $S_{2n+1}^{(1)}$ is even.

Suppose that the symmetric Hamburger moment problem is indeterminate. Then we can define the four entire functions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ given by the polynomials (S_n), $(S_n^{(1)})$ as in (5). It follows easily that $\mathscr{A}$, $\mathscr{D}$ are odd and $\mathscr{B}$, $\mathscr{C}$ are even.

Any function $\phi \in \mathscr{P}$ can be written as

$$\phi(z) = az + b + \int \frac{tz+1}{t-z}\, d\rho(t),$$

where $a \geq 0$, $b \in \mathbf{R}$ and ρ is a positive finite measure on $\mathbf{R}$. The function ϕ is uniquely determined by the triple (a, b, ρ) (see e.g. [1], p. 92). Let ϕ^* be defined by the triple $(a, -b, \check{\rho})$.

PROPOSITION 4.2. *In the Nevanlinna parametrization of the solutions to the symmetric Hamburger moment problem we have* $\check{\sigma}_\phi = \sigma_{\phi^*}$, *and* $\check{\sigma}_\gamma = \sigma_\gamma$.

Proof. We have, since $\phi(-z) = -\phi^*(z)$,

$$\begin{aligned}\int \frac{1}{t-z}\, d\check{\sigma}_\phi(t) &= -\int \frac{1}{t+z}\, d\sigma_\phi(t) \\ &= \frac{\mathscr{A}(-z)\,\phi(-z) - \mathscr{C}(-z)}{\mathscr{B}(-z)\,\phi(-z) - \mathscr{D}(-z)} \\ &= \frac{\mathscr{A}(z)\,\phi^*(z) - \mathscr{C}(z)}{-\mathscr{B}(z)\,\phi^*(z) + \mathscr{D}(z)} \\ &= \int \frac{1}{t-z}\, d\sigma_{\phi^*}(t).\end{aligned}$$

Thus $\check{\sigma}_\phi = \sigma_{\phi^*}$. In a similar way we see that $\check{\sigma}_\gamma = \sigma_\gamma$ and the proposition follows. ∎

For the Nevanlinna-extremal solutions the proposition states that $\tilde{\sigma}_t = \sigma_{-t}$, so that there are exactly two symmetric N-extremal solutions to a symmetric Hamburger moment problem. By combining Proposition 4.1 and 4.2 we see that a Stieltjes moment sequence is indet(S) if and only if the corresponding symmetric Hamburger moment sequence is indeterminate. See also Chihara, [10], regarding these two last results.

Remark 4.1. We can in fact determine all symmetric solutions to a symmetric moment problem: From Proposition 4.2 and the first line of its proof we see that σ_ϕ is symmetric if and only if ϕ is an odd function. It is readily seen that ϕ is odd if and only if

$$\phi(z) = az + \int \frac{tz+1}{t-z}\, d\rho(t),$$

where $a \geqslant 0$ and ρ is a finite symmetric measure on $\mathbf{R}$.

5. Orthogonal Polynomials

Let $t = (t_n)_{n \geqslant 0}$ be a normalized Stieltjes moment sequence and denote by $(P_n)_{n \geqslant 0}$, $(P_n^{(1)})_{n \geqslant 0}$ the associated polynomials of the first and second kind. Let $(K_n)_{n \geqslant 0}$, $(K_n^{(1)})_{n \geqslant 0}$ denote the polynomials associated with the normalized Stieltjes sequence $t' = (t_{n+1}/t_1)_{n \geqslant 0}$, and as before, let $(S_n)_{n \geqslant 0}$, $(S_n^{(1)})_{n \geqslant 0}$ denote the polynomials associated with the symmetric Hamburger moment sequence s corresponding to $(t_n)_{n \geqslant 0}$. We shall be interested in relations between these polynomials. From Chihara, [10], we have the following

Lemma 5.1. *The polynomials $(P_n)_{n \geqslant 0}$, and $(K_n)_{n \geqslant 0}$ are given by*

$$P_n(x^2) = S_{2n}(x)$$
$$xK_n(x^2) = \sqrt{t_1}\, S_{2n+1}(x).$$

We shall now determine the polynomials of the second kind:

Lemma 5.2. *The polynomials $(P_n^{(1)})_{n \geqslant 0}$, and $(K_n^{(1)})_{n \geqslant 0}$ are given by*

$$P_n^{(1)}(x^2) = \begin{cases} S_{2n}^{(1)}(x)/x, & x \neq 0 \\ S_{2n}^{(1)\prime}(0), & x = 0 \end{cases}$$

$$\begin{aligned} t_1 K_n^{(1)}(x^2) &= \sqrt{t_1}\, S_{2n+1}^{(1)}(x) - K_n(x^2) \\ &= \begin{cases} \sqrt{t_1}\, S_{2n+1}^{(1)}(x) - (\sqrt{t_1}/x)\, S_{2n+1}(x), & x \neq 0 \\ \sqrt{t_1}\, S_{2n+1}^{(1)}(0) - \sqrt{t_1}\, S'_{2n+1}(0), & x = 0. \end{cases} \end{aligned}$$

Proof. To fix notation, let μ be a solution to the symmetric moment problem and let $\sigma = \phi(\mu)$. We shall use the identities

$$\begin{aligned}(x^2-t^2)^{-1} &= (2x(x+t))^{-1} + (2x(x-t))^{-1} \\ &= (2t(x-t))^{-1} - (2t(x+t))^{-1}.\end{aligned}$$

By definition,

$$\begin{aligned}P_n^{(1)}(x^2) &= \int \frac{P_n(x^2)-P_n(t)}{x^2-t}\,d\sigma(t) \\ &= \int \frac{S_{2n}(x)-S_{2n}(t)}{x^2-t^2}\,d\mu(t) \\ &= \frac{1}{2x}\int \frac{S_{2n}(x)-S_{2n}(t)}{x+t}\,d\mu(t) + \frac{1}{2x}\int \frac{S_{2n}(x)-S_{2n}(t)}{x-t}\,d\mu(t) \\ &= -\frac{S_{2n}^{(1)}(-x)}{2x} + \frac{S_{2n}^{(1)}(x)}{2x} \\ &= \frac{S_{2n}^{(1)}(x)}{x}, \qquad x \neq 0.\end{aligned}$$

Similarly,

$$\begin{aligned}t_1 K_n^{(1)}(x^2) &= \int \frac{K_n(x^2)-K_n(t)}{x^2-t}\,t\,d\sigma(t) \\ &= \int (tK_n(x^2) - tK_n(t^2))\frac{t}{x^2-t^2}\,d\mu(t) \\ &= \frac{1}{2}\int \frac{tK_n(x^2)-tK_n(t^2)}{x-t}\,d\mu(t) - \frac{1}{2}\int \frac{tK_n(x^2)-tK_n(t^2)}{x+t}\,d\mu(t) \\ &= -\frac{1}{2}K_n(x^2) + \frac{\sqrt{t_1}}{2}S_{2n+1}^{(1)}(x) - \frac{1}{2}K_n(x^2) + \frac{\sqrt{t_1}}{2}S_{2n+1}^{(1)}(-x) \\ &= \sqrt{t_1}\,S_{2n+1}^{(1)}(x) - K_n(x^2),\end{aligned}$$

which implies the lemma. ∎

As an immediate consequence of Lemma 5.1 we have

PROPOSITION 5.1. *The functions $p_s, p_t, p_{t'}$ corresponding to the three sequences (cf. (3)), satisfy*

$$p_t(z^2)^2 + \frac{|z|^2}{t_1} p_{t'}(z^2)^2 = p_s(z)^2. \tag{13}$$

COROLLARY 5.1 [8, Proposition 2.3]. *The Stieltjes sequence t is indet(S) if and only if t and t' are both indeterminate.*

Proof. If t and t' are both indeterminate, (13) implies that $p_s(z)$ is finite for some non-real z. Therefore s is indeterminate and so t is indet(S) as noted in Section 1. ∎

6. THE NEVANLINNA MATRICES OF ENTIRE FUNCTIONS

Suppose that the Stieltjes sequence $(t_n)_{n \geq 0}$ is indet(S). In this case the corresponding symmetric Hamburger moment problem is indeterminate and so is the sequence $(t_{n+1}/t_1)_{n \geq 0}$. We shall find relations between the entire functions associated with these three indeterminate moment problems.

Denote by A, B, C, D the functions associated with the sequence $(t_n)_{n \geq 0}$, by A_1, B_1, C_1, D_1 those associated with $(t_{n+1}/t_1)_{n \geq 0}$ and as before $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ those associated with the symmetric problem.

We shall make use of the two-variable analogues of the entire functions $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$, see Buchwalter & Cassier, [6]. By differentiating these functions with respect to the first variable and by using their Proposition 4.6, p. 177 we obtain the following identities:

LEMMA 6.1.

$$z \sum S_n^{(1)\prime}(0)\, S_n^{(1)}(z) = -\mathscr{A}'(0)\, \mathscr{C}(z) + \frac{1}{z} \mathscr{A}(z)$$

$$z \sum S_n^{(1)\prime}(0)\, S_n(z) = -\mathscr{A}'(0)\, \mathscr{D}(z) + \frac{1}{z} (\mathscr{B}(z) + 1)$$

$$z \sum S_n'(0)\, S_n^{(1)}(z) = \mathscr{D}'(0)\, \mathscr{A}(z) + \frac{1}{z} (\mathscr{C}(z) - 1)$$

$$z \sum S_n'(0)\, S_n(z) = \mathscr{D}'(0)\, \mathscr{B}(z) + \frac{1}{z} \mathscr{D}(z).$$

We can now formulate

Proposition 6.1. *The functions A, B, C, D are given in terms of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$*

$$\begin{aligned}
A(z^2) &= -\mathcal{A}'(0)\,\mathcal{C}(z) + \mathcal{A}(z)/z \\
B(z^2) &= \mathcal{B}(z) - \mathcal{A}'(0)\,z\mathcal{D}(z) \\
C(z^2) &= \mathcal{C}(z) \\
D(z^2) &= z\mathcal{D}(z).
\end{aligned}$$

Remark 6.1. These formulas are derived in Chihara, [10], using methods like the one referred to in Section 3. We note that if $\alpha < 0$ it may be written as

$$\alpha = -\frac{1}{\mathcal{A}'(0)}.$$

This follows by combining the expressions for B and D in the above proposition and comparing with (3.17) in Chihara, [10]. We give a direct proof:

From Akhiezer, [1], p. 14 we have

$$\begin{aligned}
\mathcal{A}_{2n}(x) &= b_{2n-1}(S^{(1)}_{2n-1}(0)\,S^{(1)}_{2n}(x) - S^{(1)}_{2n}(0)\,S^{(1)}_{2n-1}(x)) \\
&= b_{2n-1}\,S^{(1)}_{2n-1}(0)\,S^{(1)}_{2n}(x)
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{A}'_{2n}(0) &= b_{2n-1}\,S^{(1)}_{2n-1}(0)\,S^{(1)\prime}_{2n}(0) \\
&= b_{2n-1}\,S^{(1)}_{2n-1}(0)\,S_{2n}(0)\,\frac{S^{(1)\prime}_{2n}(0)}{S_{2n}(0)} \\
&= -\frac{S^{(1)\prime}_{2n}(0)}{S_{2n}(0)},
\end{aligned}$$

by Akhiezer, [1], p. 9. Lemma 5.2 then gives

$$\alpha = \lim_{n\to\infty} \frac{P_n(0)}{P^{(1)}_n(0)} = \lim_{n\to\infty} \frac{S_{2n}(0)}{S^{(1)\prime}_{2n}(0)} = -\lim_{n\to\infty} \frac{1}{\mathcal{A}'_{2n}(0)} = -\frac{1}{\mathcal{A}'(0)}.$$

Since $\mathcal{A}'_{2n}(0)$ increases with n we get a lower bound for α:

$$\alpha \geqslant \frac{P_1(0)}{P^{(1)}_1(0)} = -t_1.$$

The proof of Proposition 6.1 is straightforward using the Definitions (5) and the Lemmas 5.1, 5.2, 6.1.

PROPOSITION 6.2. *The functions A_1, B_1, C_1, D_1 can in terms of $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ be expressed as*

$$A_1(z^2)=\frac{1}{t_1}\left((z\mathscr{A}(z)-\mathscr{B}(z))(1-\mathscr{D}'(0))-\mathscr{C}(z)+\frac{1}{z}\mathscr{D}(z)\right)$$

$$B_1(z^2)=\mathscr{B}(z)(1-\mathscr{D}'(0))-\frac{1}{z}\mathscr{D}(z)$$

$$C_1(z^2)=(z\mathscr{A}(z)-\mathscr{B}(z))\,\mathscr{D}'(0)+\mathscr{C}(z)-\frac{1}{z}\mathscr{D}(z)$$

$$D_1(z^2)=t_1\left(\mathscr{B}(z)\,\mathscr{D}'(0)+\frac{1}{z}\mathscr{D}(z)\right).$$

The proof follows the same computational lines as indicated above. To illustrate we compute $D_1(z^2)$:

$$\begin{aligned} D_1(z^2)&=z^2\sum K_n(0)\,K_n(z^2)\\ &=t_1 z\sum S'_{2n+1}(0)\,S_{2n+1}(z)\\ &=t_1\left(\mathscr{B}(z)\,\mathscr{D}'(0)+\frac{1}{z}\mathscr{D}(z)\right). \end{aligned}$$

By (10), $\mathscr{A}'(0)=-1/\alpha=-\lim_{x\to-\infty}B(x)/D(x)$. The number $\mathscr{D}'(0)$ can be written as $D'(0)$. Thus combining Proposition 6.1 and 6.2 we obtain:

PROPOSITION 6.3. *With $\alpha=\lim_{x\to-\infty}D(x)/B(x)$ we have*

$$\begin{aligned} A_1(z)=\frac{1}{t_1}\Bigg(&zA(z)(1-D'(0))-B(z)(1-D'(0))\\ &-C(z)\left(\frac{z}{\alpha}(1-D'(0))+1\right)+D(z)\left((1-D'(0))\frac{1}{\alpha}+\frac{1}{z}\right)\Bigg) \end{aligned}$$

$$B_1(z)=B(z)(1-D'(0))-D(z)\left(\frac{1}{\alpha}(1-D'(0))+\frac{1}{z}\right)$$

$$C_1(z) = zA(z)\,D'(0) - B(z)\,D'(0) + C(z)\left(-\frac{z}{\alpha}D'(0) + 1\right)$$
$$+ D(z)\left(\frac{1}{\alpha}D'(0) - \frac{1}{z}\right)$$
$$D_1(z) = t_1\left(B(z)\,D'(0) + D(z)\left(-\frac{1}{\alpha}D'(0) + \frac{1}{z}\right)\right).$$

We can write the conclusion of Proposition 6.3 in matrix form as

$$\begin{pmatrix} A_1(z) \\ B_1(z) \\ C_1(z) \\ D_1(z) \end{pmatrix} = N(\alpha, D'(0)) \begin{pmatrix} A(z) \\ B(z) \\ C(z) \\ D(z) \end{pmatrix},$$

where $N(\alpha, D'(0))$ is the matrix

$$\begin{pmatrix} \frac{z}{t_1}(1-D'(0)) & -\frac{1}{t_1}(1-D'(0)) & -\frac{z}{t_1\alpha}(1-D'(0)) - \frac{1}{t_1} & \frac{1}{t_1\alpha}(1-D'(0)) + \frac{1}{t_1 z} \\ 0 & (1-D'(0)) & 0 & -\frac{1}{\alpha}(1-D'(0)) - \frac{1}{z} \\ zD'(0) & -D'(0) & -\frac{z}{\alpha}D'(0)+1 & \frac{1}{\alpha}D'(0) - \frac{1}{z} \\ 0 & t_1 D'(0) & 0 & -\frac{t_1}{\alpha}D'(0) + \frac{t_1}{z} \end{pmatrix}.$$

By iteration we get

PROPOSITION 6.4. *If $(t_n)_{n \geq 0}$ is a Stieltjes moment sequence which is indet(S), the entire functions associated with the sequence $(t_{n+k}/t_k)_{n \geq 0}$, A_k, B_k, C_k, D_k, can be expressed in terms of those associated with $(t_n)_{n \geq 0}$:*

$$\begin{pmatrix} A_k(z) \\ B_k(z) \\ C_k(z) \\ D_k(z) \end{pmatrix} = N(\alpha_{k-1}, D'_{k-1}(0)) \cdot \cdots \cdot N(\alpha, D'(0)) \begin{pmatrix} A(z) \\ B(z) \\ C(z) \\ D(z) \end{pmatrix},$$

with $\alpha_i = \lim_{x \to -\infty} D_i(x)/B_i(x)$.

We are in fact able to describe the N-matrix corresponding to the measure $p(x)\,d\sigma(x)$ in terms of the functions in the N-matrix corresponding to σ, where p is any polynomial with positive leading coefficient and having only real non-positive zeros. The key to this is Propositions 3.3, 6.3 and the following

LEMMA 6.2. *When $c \geqslant 0$, the measure $x\, d\tau_{-c}(\sigma)(x)$ is concentrated on the positive axis and*

$$(x+c)\, d\sigma(x) = \tau_c(x\, d\tau_{-c}(\sigma)(x)).$$

We shall not give explicit formulas.

We are now going to investigate the growth of the entire functions considered in this section. For an indeterminate Hamburger moment problem the four entire functions given in (5) are of at most zero exponential type, see [1], p. 56 and it turns out that the four functions all have the same order, type and Phragmén–Lindelöf indicator function, see Berg & Pedersen, [4]. Concerning the notion of order, type and indicator function see e.g. Markushevich, [14].

PROPOSITION 6.5. *Let t be a normalized Stieltjes moment sequence which is indet(S) and let s be the corresponding symmetric Hamburger moment sequence. If we denote by (ρ_t, τ_t, h_t) (resp. (ρ_s, τ_s, h_s)) the order, type and indicator of the functions associated with t (resp. s) then*

$$\rho_t = \tfrac{1}{2}\rho_s, \qquad \tau_t = \tau_s, \qquad h_t(2\theta) = h_s(\theta).$$

Proof. From Proposition 6.1 we have $C(z^2) = \mathscr{C}(z)$ which immediately yields the result. ∎

PROPOSITION 6.6. *The order of the entire functions associated with a Stieltjes moment problem which is indet(S) is less than or equal to $\frac{1}{2}$.*

Concerning the relation between the growth of the functions A, B, C, D and A_k, B_k, C_k, D_k, $k \geqslant 1$ we have

PROPOSITION 6.7. *The order, type and indicator function of the functions associated with $(t_{n+k}/t_k)_{n \geqslant 0}$ are the same as the order, type and indicator function of the functions associated with $(t_n)_{n \geqslant 0}$.*

Proof. The matrix $N(\alpha, D'(0))$ is non-singular for $z \neq 0$ (the determinant is in fact 1 for all $z \neq 0$). The functions A, B, C, D are thus given uniquely in terms of A_1, B_1, C_1, D_1. The entries of $N(\alpha, D'(0))^{-1}$ are as well rational functions of z. The result follows from this observation. ∎

7. The Nevanlinna Parametrizations

We shall relate the Nevanlinna parametrization of the solutions to the problem $(t_{n+1}/t_1)_{n\geq 0}$ to the solutions to the problem $(t_n)_{n\geq 0}$. We use the superscript (0) (resp. (1)) to indicate if a measure is a solution to $(t_n)_{n\geq 0}$ (resp. to $(t_{n+1}/t_1)_{n\geq 0}$). For any solution on the positive axis $\sigma_f^{(0)}$, $(t/t_1)\,d\sigma_f^{(0)}$ is clearly a solution to the moment problem $(t_{n+1}/t_1)_{n\geq 0}$ and is therefore of the form $\sigma_g^{(1)}$ for some $g\in\mathscr{P}\cup\{\infty\}$. We shall determine the g's thus obtained, i.e. we shall describe the partially defined mapping $T: f\mapsto g$ of $\mathscr{P}\cup\{\infty\}$ into $\mathscr{P}\cup\{\infty\}$.

PROPOSITION 7.1. *The mapping T is given as*

$$T(f)(z) = -t_1\left(1-\frac{z(1-f(z)/\alpha)}{f(z)+zD'(0)(1-f(z)/\alpha)}\right)^{-1}.$$

Proof. Let $h\in\mathscr{P}\cup\{\infty\}$. A tiresome calculation shows that

$$\frac{A_1(z)\,h(z)-C_1(z)}{B_1(z)\,h(z)-D_1(z)}=\frac{1}{t_1}\left(-1+\frac{A(z)\,h^\star(z)-C(z)}{B(z)\,h^\star(z)-D(z)}\right),$$

where $h^\star$ is given by

$$h^\star(z)=\left(\frac{1}{\alpha}+\frac{1}{z}\,\frac{h(z)+t_1}{h(z)(1-D'(0))-t_1D'(0)}\right)^{-1}.$$

As mentioned above, for $f\in\mathscr{P}\cup\{\infty\}$,

$$\frac{t}{t_1}\,d\sigma_f^{(0)}(t)=d\sigma_g^{(1)}(t),$$

for exactly one g in $\mathscr{P}\cup\{\infty\}$. Now,

$$\frac{1}{t_1}\int\frac{1}{z-t}\,t\,d\sigma_f^{(0)}(t)=\frac{1}{t_1}\left(-1+z\int\frac{1}{z-t}\,d\sigma_f^{(0)}(t)\right)$$

$$\int\frac{1}{z-t}\,d\sigma_g^{(1)}(t)=\frac{1}{t_1}\left(-1+z\,\frac{A(z)\,g^\star(z)-C(z)}{B(z)\,g^\star(z)-D(z)}\right),$$

so we must have $g^\star(z)=f(z)$. Solving this equation with respect to g gives

$$g(z)=-t_1\left(1-\frac{z(1-f(z)/\alpha)}{f(z)+zD'(0)(1-f(z)/\alpha)}\right)^{-1}. \quad\blacksquare$$

The N-extremal solutions to the Stieltjes moment problem $(t_n)_{n\geq 0}$ are as mentioned above given by $\sigma_t^{(0)}$, $t\in[\alpha;0]$. In Berg & Thill, [8], a density index for these measures is defined; it is the largest integer $k \geq 0$ such that the polynomials are dense in the space $L^2(x^k\, d\sigma_t^{(0)}(x))$ and it is denoted $\delta(\sigma_t)$. It is proved that

$$\delta(\sigma_t) = \begin{cases} 1 & t = \alpha \\ 0 & \alpha < t < 0 \\ 2 & t = 0. \end{cases} \tag{14}$$

This can also be seen by Proposition 7.1. We get that

$$\frac{t}{t_1}\, d\sigma_\alpha^{(0)}(t) = d\sigma_{-t_1}^{(1)}(t),$$

$$\frac{t}{t_1}\, d\sigma_0^{(0)}(t) = d\sigma_\beta^{(1)}(t),$$

where $\beta = -t_1(1-(1/D'(0)))^{-1} \in \mathbf{R}$. Applying our proposition again we see that

$$\frac{t^2}{t_2}\, d\sigma_\alpha^{(0)}(t) = d\sigma_h^{(2)}(t),$$

$$\frac{t^2}{t_2}\, d\sigma_0^{(0)}(t) = d\sigma_\gamma^{(2)}(t),$$

with $\gamma \in \mathbf{R}$ and h a certain rational function of degree 1. Finally, applying the proposition once again, we get that

$$\frac{t^3}{t_3}\, d\sigma_0^{(0)}(t) = d\sigma_p^{(3)}(t),$$

with p a certain rational function of degree 1. This gives another (and I must admit) much more technical proof of (14).

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